Mathl Modelling, Vol. 9, No. 8, pp. 573–576, 1987

0270-0255/87 $3.00 + 0.00


# OPTIMAL DUE-DATE DETERMINATION AND SEQUENCING WITH RANDOM PROCESSING TIMES

T. C. E. CHENG

Department of Actuarial and Management Sciences, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada

(*Received February* 1987)

Communicated by E. Y. Rodin

**Abstract**—This paper considers the problem of due-date determination and sequencing of $n$ stochastically independent jobs on a single machine with random processing times. The objective is to find the optimal due-date values for the constant due-date assignment method and the optimal job sequence that minimize the expected value of a total cost function. It is shown that under suitable assumptions the optimal due-date values can be analytically determined and the jobs should be arranged in the SEPT sequence to minimize the cost.

## INTRODUCTION

Scheduling against due-date has been a popular research topic over the last twenty years. Conway [1] was probably the first researcher to conduct a systematic study on due-date assignment methods in scheduling. His approach uses computer simulation to investigate the relative effectiveness of various due-date assignment rules under a variety of simulated job shop conditions. Conway's results have induced considerable interest in due-date scheduling research and computer simulation has become a standard tool for analysis in the majority of subsequent studies. Examples of due-date scheduling research based on computer simulation are, among others, Cheng [2], Eilon and Chowdhury [3], Elvers [4], Jones [5], Rochette and Sadowski [6] and Weeks and Fryer [7].

In recent years there has been growing interest among scheduling researchers in seeking optimal due-date assignment policies. Since computer simulation is essentially a trial-and-error method, it is not suited for the optimal due-date determination problems. A new breed of researchers has emerged who propose to tackle the optimal due-date assignment problems analytically. The general approach of these studies is to develop a mathematical model of the due-date determination problem and analytically solve for the optimal result that minimizes some pre-defined due-date related performance measures. Among researchers who have made contributions in this area are Cheng [8,9], Kanet [10], Panwalker *et al.* [11], Seidmann *et al.* [12] and Seidmann and Smith [13].

A common assumption found in the majority of the above studies is that the job processing times are deterministic and known before processing starts. To date it seems that research directed toward seeking optimal solutions to due-date assignment problems in which job processing times are random variables has largely been ignored. The exception, perhaps, is Cheng [14].

In this paper we present a model to study the optimal due-date determination problem with random processing times. A simple due-date assignment method is used which assigns a constant flow allowance to the jobs, which are not necessarily the same. The objective is to determine the optimal values of these constant flow allowances and the optimal job sequence so that the expected total cost of due-date assignment and missing due-dates is minimized.

## PROBLEM FORMULATION

In this study we are concerned with sequencing $n$ independent jobs on a single machine. Let $N$ be the set of jobs each requiring $p_i$ amount of processing time on the machine. It is assumed that the job processing times $p_i$ are random variables with mean $\mu_i$ and standard deviation $\sigma_i$, $\forall i \in N$. The jobs are available for processing at the same time. The machine cannot simultaneously process more than one job and no job splitting is permitted.

The constant due-date assignment method is used to assign due-dates to the jobs, i.e. for job $i$ $d_i = k_i$, where $k_i$ is a constant flow allowance, $\forall i \in N$. Let $S$ be the set of permutations of the $n$ jobs and $s$ be anyone of the $n!$ possible job sequences. If the subscript $[i]$ denotes the job in position $i$ of $s$, then $E_{[i]}$, $L_{[i]}$ and $C_{[i]}$, respectively, denote the earliness, tardiness and completion time of job $[i]$ in $s$, $1 \leqslant i \leqslant n$.

For the due-date determination and sequencing problem, two types of cost are of relevance; namely, (1) the cost associated with the assigned due-date values and (2) the cost of missing due-dates. Both these types of cost are opportunity costs because reduction of these costs would mean possible gain in intangible benefits, We assume that, for a job sequence $s$, the individual due-date cost for job $i$ is $\phi(k_i \mid s)$ and the total due-date cost is $\sum_{1 \leqslant i \leqslant n} \phi(k_i \mid s)$. The cost of missing due-dates is assumed to be a function of the squared value of the difference between the completion time and due-date of a job, i.e. $\theta((C_i - k_i \mid s)^2)$, and so the total cost of missing due-dates is $\sum_{1 \leqslant i \leqslant n} \theta((C_i - k_i \mid s)^2)$. The total cost is thus the sum of these two components whose expected value to be minimized can be written as

$$E(C(\mathbf{k} \mid s)) = \sum_{1 \leqslant i \leqslant n} \phi(k_i \mid s) + \sum_{1 \leqslant i \leqslant n} \int_0^\infty \theta((C_i - k_i \mid s)^2) f(C_i) \, \mathrm{d}C_i, \tag{1}$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ is a vector of the constant flow allowances, $f(C_i)$ is the probability density function (pdf) of the completion time of job $i$ and $E(\cdot)$ is the expected value operator.

Further assumptions about the cost functions $\phi(k_i \mid s)$ and $\theta((C_i - k_i \mid s)^2)$, $1 \leqslant i \leqslant n$, are that they are both monotone increasing, convex and twice continuously differentiable such that they vanish at the origin. Justification of the convexity assumption is found in Jones [5], who reports that both the due-date assignment cost and missing due-date cost exhibit exponential growth. The other assumptions are standard assumptions normally made to facilitate the mathematical analysis.

## OPTIMAL DUE-DATES

Differentiating equation (1) with respect to $\mathbf{k}$ once and twice respectively, we obtain the first and second partial derivatives of $E(C(\mathbf{k} \mid s))$ as follows:

$$\partial E(C(\mathbf{k} \mid s))/\partial k_i = \phi' + 2 \int_0^\infty (k_i - C_i)\theta' f(C_i) \, \mathrm{d}C_i, \quad 1 \leqslant i \leqslant n; \tag{2}$$

and

$$\partial^2 E(C(\mathbf{k} \mid s))/\partial k_i^2 = \phi'' + 4 \int_0^\infty (k_i - C_i)^2 \theta'' f(C_i) \, \mathrm{d}C_i + 2 \int_0^\infty \theta' f(C_i) \, \mathrm{d}C_i, \quad 1 \leqslant i \leqslant n. \tag{3}$$

Let $\mathbf{k}^* = (k_1^*, k_2^*, \ldots, k_n^*)$ be a vector of due-date allowances that makes the first partial derivatives of $E(C(\mathbf{k} \mid s))$ vanish. Then

$$\partial E(C(\mathbf{k}^* \mid s))/\partial k_i = \phi' + 2 \int_0^\infty (k_i^* - C_i)\theta' f(C_i) \, \mathrm{d}C_i = 0, \quad 1 \leqslant i \leqslant n. \tag{4}$$

Since $\partial^2 E(C(\mathbf{k} \mid s))/\partial k_i^2 \geqslant 0$ and $\partial^2 E(C(\mathbf{k} \mid s))/\partial k_i \partial k_j = 0$, for $\mathbf{k} = \mathbf{k}^*$, then it is easily verified that the Hessian matrix of $E(C(\mathbf{k} \mid s))$ is positive semidefinite. Therefore, $\mathbf{k}^*$ is an optimal point that minimizes $E(C(\mathbf{k} \mid s))$. In addition, it has been assumed that both $\phi$ and $\theta$ are convex functions, so $E(C(\mathbf{k} \mid s))$ is also convex. It follows that any minimum point of $E(C(\mathbf{k} \mid s))$ is also an absolute minimum point. Thus $\mathbf{k}^*$ is an absolute minimum point of $E(C(\mathbf{k} \mid s))$.

## LINEAR COST MODEL

We will now consider a linear cost model of the due-date determination and sequencing problem. Among researchers who have discussed the use of linear cost functions in scheduling problems are Jones [5] and Eilon and Chowdhury [3], to name a few. In fact, in real-life situations it is usually possible to obtain a reasonably accurate linear approximation of the total cost function.

In the linear cost model, the due-date assignment cost and the missed due-date cost are, respectively, assumed to be as follows:

$$\phi(k_i \mid s) = \alpha(k_i \mid s) \tag{5}$$

and

$$\theta((C_i - k_i \mid s)^2) = \beta(C_i - k_i \mid s)^2, \tag{6}$$

where $\alpha$ and $\beta$ are known positive real constants.

Substituting equations (5) and (6) into equation (1) yields the following expected total cost:

$$E(C(\mathbf{k} \mid s)) = \sum_{1 \leqslant i \leqslant n} \alpha(k_i \mid s) + \sum_{1 \leqslant i \leqslant n} \int_0^{\infty} \beta(C_i - k_i \mid s)^2 f(C_i)\,\mathrm{d}C_i. \tag{7}$$

Differentiating equation (7) with respect to $k_i$ once, $1 \leqslant i \leqslant n$, equating the result to zero and solving for $k_i$, we obtain

$$(k_i^* \mid s) = \beta E(C_i \mid s) - \alpha/2$$

or, using the subscript $[i]$ to denote the job in position $i$ of job sequence $s$,

$$k_{[i]}^* = \beta \sum_{1 \leqslant j \leqslant i} \mu_{[j]} - \alpha/2. \tag{8}$$

Substituting equation (8) into equation (7) and simplifying yields the following minimum expected total cost:

$$\begin{aligned} E(C(\mathbf{k}^* \mid s)) &= \alpha\beta \sum_{1 \leqslant i \leqslant n} \sum_{1 \leqslant j \leqslant i} \mu_{[j]} - n\alpha^2/2 \\ &\quad + \sum_{1 \leqslant i \leqslant n} \beta \left\{ \sum_{1 \leqslant j \leqslant i} [\sigma_{[j]}^2 + (\beta - 1)^2 \mu_{[j]}^2 + \alpha(\beta - 1)\mu_{[j]}] + \alpha^2/4 \right\} \\ &= \sum_{1 \leqslant i \leqslant n} (n - i + 1)[\alpha\beta^2 \mu_{[i]} + \beta\sigma_{[i]}^2 + \beta(\beta - 1)^2 \mu_{[i]}^2] + n\alpha^2(\beta - 2)/4. \end{aligned} \tag{9}$$

## OPTIMAL JOB SEQUENCE

It is easily seen that equation (9) is equal to the sum of the products of two non-negative sequences plus a constant. In general, it is a hard combinatorial problem to find an optimal job sequence $s^*$ to minimize equation (9). However, an elegant solution method based on a well-known mathematical theorem can be devised to determine the optimal job sequence if one of the following conditions holds:

(A) All job processing times have equal standard deviation, i.e. $\sigma_i = \sigma_j$, $\forall i, j \in N$.
(B) The standard deviations of job processing times are monotonically increasing functions of the mean processing times, i.e. $\sigma_i = \gamma(\mu_i)$, where $\gamma'(\cdot) \geqslant 0$, $\forall i \in N$.

It is a well-known mathematical theorem that the minimum of the sum of the products of two non-negative sequences is obtained by arranging one sequence in ascending order and the other in descending order (see Hardy [15, p. 262]). Under one of the above conditions, $E(C(\mathbf{k}^* \mid s))$ is the sum of the products of a sequence of positional penalty indexes and another sequence with values determined by the mean job processing times plus a constant. Thus we can construct an optimal job sequence $s^*$ that minimizes equation (9) by assigning the job with the smallest mean processing time to the first position, the job with the second smallest mean processing time to the second position and so on. This is because the positional penalty index sequence is decreasing and so the corresponding mean job processing time sequence must be increasing, i.e. $s^*$ should be in the shortest-expected-processing-time (SEPT) sequence.

The two conditions stated above may not be as restrictive as they first appear. Some commonly assumed processing time distributions, such as the exponential and gamma distributions, satisfy the second condition quite readily. It is easily verified that for an exponential distribution $\sigma = \mu$, while for a gamma distribution $\sigma = \mu/\sqrt{a}$, where $a$ is a parameter of the gamma distribution. Furthermore, under certain circumstances, the assumption that all jobs have the same standard deviations can be made, then the first condition is satisfied and the simple solution procedure to determine the optimal sequence is still applicable.

## CONCLUSION

This paper examines the problem of due-date determination and sequencing of $n$ jobs on a single machine with random processing times. The objective is to find the optimal due-date values and job sequence that minimize the expected value of a total cost function. It has been shown that under suitable assumptions the optimal due-date values can be analytically determined and the jobs should be arranged in the SEPT sequence.

## REFERENCES

1. R. W. Conway, Priority dispatching and job lateness in a job shop. *J. ind. Engng* **16**, 228–237 (1965).
2. T. C. E. Cheng, Optimal due-date assignment in a job shop. *Int. J. Product. Res.* **24**, 503–515 (1986).
3. S. Eilon and I. G. Chowdhury, Due-date in job shop scheduling, *Int. J. Product. Res.* **14**, 223–238 (1976).
4. D. A. Elvers, Job shop dispatching rules using various delivery date setting criteria. *Product. Inventory Mgmt* **14**, 62–70 (1973).
5. C. H. Jones, An economic evaluation of job shop dispatching rules. *Mgmt Sci.* **20**, 293–303 (1973).
6. R. Rochette and R. P. Sadowski, A statistical comparison of the performance of simple dispatching rules for a particular set of job shops. *Int. J. Product. Res.* **14**, 63–70 (1976).
7. J. K. Weeks and J. S. Fryer, A simulation study of operating policies in a hypothetical dual-constrained job shop. *Mgmt Sci.* **22**, 1362–1371 (1976).
8. T. C. E. Cheng, Optimal due-date determination and sequencing of $n$ jobs on a single machine. *J. opl Res. Soc.* **35**, 433–437 (1984).
9. T. C. E. Cheng, Analytical determination of optimal TWK due-date in a job shop. *Int. J. Syst. Sci.* **16**, 777–787 (1985).
10. J. J. Kanet, Minimizing the average deviation of job completion times about a common due-date. *Naval Res. Logist. Q.* **28**, 634–651 (1981).
11. S. S. Panwalker, M. L. Smith and A. Seidmann, Common due-date assignment to minimize total penalty for the one machine scheduling problem. *Opns Res.*, **30**, 391–399 (1982).
12. A. Seidmann, S. S. Panwalker and M. L. Smith, Optimal assignment of due-dates for a single processor scheduling problem. *Int. J. Product. Res.* **19**, 393–399 (1981).
13. A. Seidmann and M. L. Smith, Due-date assignment for production systems. *Mgmt Sci.* **27**, 571–581 (1981).
14. T. C. E. Cheng, Optimal due-date assignment for a single machine sequencing problem with random processing times. *Int. J. Syst. Sci.* **17**, 1139–1144 (1986).
15. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*. Cambridge Univ. Press, Cambridge (1952).